Automated Reasoning Tools in GeoGebra

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http://www.recio.tk
Summary:

• What’s new: cooperation GeoGebra + ART

• Demo session

• Behind the scene

• Towards an intelligent geometry book
What is GeoGebra?


• GeoGebra is dynamic mathematics software for all levels of education that brings together geometry, algebra, spreadsheets, graphing, statistics and calculus in one easy-to-use package.

• In 2013, Bernard Parisse's Giac was integrated into GeoGebra's CAS view.

• GeoGebra is a rapidly expanding community of about forty millions users, located in just about every country. Available in many languages. GeoGebra Materials: 1 million resources (April 2016)

• Desktop, web, tablet, smartphone versions.

• Open source software freely available for non-commercial users

• https://en.wikipedia.org/wiki/GeoGebra
1. Solve\(\{x^2+2xy-3, x-2y+z^2\}, \{x,y,z\}\)
   \[\rightarrow \left\{\left\{x = x, y = -\frac{1}{2} x^2 + \frac{3}{2}, z = \sqrt{-x^2 - x + 3}\right\}, \left\{x = x, y = -\frac{1}{2} x^2 + \frac{3}{2}, z = -\sqrt{-x^2 - x + 3}\right\}\right\}\]

2. Eliminate\(\{x^2+2xy-3, x-2y+z^2, x^3-3y\}, \{z\}\)
   \[\rightarrow \{x^2 + 2y - 3, 2x - 3y + 3x + 3y, 8y^2 - 9x - 15y + 18\}\]

3. Integral\(e^{x^2}\)
   \[\rightarrow -\frac{1}{2} \sqrt{\pi} \cdot \frac{\text{erf}(-x \sqrt{-\ln(e)})}{\sqrt{-\ln(e)}} + c_1\]

4. Solutions\(e^{x} = 1, x\)
   \[\rightarrow \{0\}\]

5. Factor\(x^49 - 1\)
   \[\rightarrow (x - 1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)(x^{42} + x^{35} + x^{28} + x^{21} + x^{14} + x^7 + 1)\]

6. Algebra
   - Implicit Curve
     - a: \(-x^4 + x^2 + y^3 = 0\)
GeoGebra ART: Automated reasoning tools

• Automated derivation
• Automated proving
• Automated discovery
• Locus: mover-tracer, boolean, envelopes, etc.

F. Botana, Universidad de Vigo; Z. Kovács, The Private University College of Education of the Diocese of Linz; T. R.
• https://www.researchgate.net/project/Theorem-proving-tools-in-GeoGebra


• DERIVATION
• DERIVATION

Input: Relation[g, h]
• PROVE
• DISCOVERY
• DISCOVERY

Implicit Curve
- \( d: 20000x^2 - 60000x + 2 \)

Point
- \( A = (0, 0) \)
- \( B = (3, 0) \)
- \( C = (1.08, 2) \)
- \( D = (1.31, 0.79) \)
- \( D' = (2.28, 1.72) \)
- \( D'_1 = (-0.07, 1.53) \)
- \( D'_2 = (1.31, -0.79) \)

Segment
- \( a = 2.78 \)
- \( b = 2.27 \)
- \( c = 3 \)

Triangle
- \( \text{poly1} = 3 \)
• DISCOVERY

Input: ProveDetails[ArcCollinear[D', D'_1, D'_2]]
Conic
- \(d: (x - 2.91)^2 + (y - 3.25)^2 = 3.81\)

List
- `list1 = {true, "AreEqual[A,B]", "AreEqual[A,C]", "AreEqual[B,C]"}`

Point
- \(A = (1.28, 2.18)\)
- \(B = (4.26, 1.84)\)
- \(C = (3.78, 5)\)
- \(D = (1.47, 4.57)\)
- \(D' = (3.63, 2.66)\)
- \(D'_1 = (6.11, 5.27)\)
- \(D'_2 = (0.93, -0.19)\)

Segment
- \(a = 3.2\)
- \(b = 3.77\)
- \(c = 3\)

Triangle
- \(\text{poly1} = 4.63\)
Implicit Curve
- \( a: 6250000000000000x^4 \)
- \( b: 6250000000000000x^4 \)

Line
- \( f: -4.82x + 4.9y = 0 \)
- \( g: 4.82x + 1.1y = 28.92 \)
- \( h: y = 0 \)
- \( i: -4.9x - 4.82y = -8.37 \)
- \( j: -1.1x + 4.82y = -1.88 \)

Number
- \( \text{distanceDF} = 4.18 \)
- \( \text{distanceED} = 1.2 \)

Point
- \( A = (0, 0) \)
- \( B = (4.9, 4.82) \)
- \( C = (6, 0) \)
- \( D = (1.71, 0) \)
- \( E = (0.87, 0.85) \)
- \( F = (5.79, 0.93) \)

Text
- \( \text{TextDF} = \text{"DF} = 4.18\" \)
- \( \text{TextED} = \text{"ED} = 1.2\" \)
A = (7.72, 3.66)
B = (2.68, -2.6)
C = (8.92, -2.24)
b = 6.02
a = 6.25
c = 8.04
poly1 = 18.62
D = (5.75, 1.21)
E = (8.32, 0.71)
f: 0.5x + 2.57y = 6.02
d: 10000x^2 - 104000x + 10000y^2
G = ortocentro
H = circumcentro
K = baricentro
L = incentro
Behind the scene

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<tr>
<td></td>
<td>((H, T))</td>
<td>((H, T^*z-1))</td>
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<tr>
<td>not gen.true and not gen. false</td>
<td>((0))</td>
<td>((0))</td>
</tr>
<tr>
<td>generally true (and, thus, not generally false)</td>
<td>((0))</td>
<td>(\text{Not}(0))</td>
</tr>
<tr>
<td>generally false (and, thus, not generally true)</td>
<td>(\text{Not}(0))</td>
<td>((0))</td>
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ATP in DG by checking a number of instances

Proof by exhaustion, also known as proof by cases... is a method...in which the statement to be proved is split into a finite number of cases and each case is checked to see if the proposition in question holds (Wikipedia)

Example: sum of first n natural numbers S(n).
Assume S(n) polynomial, degree at most 2.

\[n=0, \ n=1, \ n=2, \ n(n+1)/2\]

(C. McBride, Calculemus 2012, notes by JHD)
• We have to establish a procedure for deciding, given a polynomial ideal (of hypotheses and [negated] theses), whether the result of eliminating in the ideal some variables, yields the zero ideal or not.

• Geometric Degree, Bezout’s Th.
• If we have a bound on

the degree d of the polynomials in the ideal,
the number n of variables and/or
the number of polynomials s,

then there is a bound D (d^s or d^n) on the degree of the variety defined by the ideal and, thus, a bound on the elimination variety, which is either zero or contained in a hypersurface of degree bounded by D.
• The elimination is not zero
  iff
its zero set is contained in a hypersurface of degree bounded by $D$
  iff
the projection is contained in a hypersurface of degree bounded by $D$
  iff
the statement is generally true.

Similarly, for the generally false case.
Procedure

• Calculate an upper bound $D$ for the degree of the zero set of hypotheses and negated thesis.

• Create a $D$-suitable collection of test points on the affine space of independent variables and test for each point of this collection if it is contained in the projection.

• If so, then the projection is the whole space and the elimination ideal is zero.

• Else, the projection is obviously not the whole space and thus the elimination is not the zero ideal.
Intersection of three altitudes

• Given a triangle $A(0,0), B(1,0), C(r, s)$, each altitude equation is linear in $(x, y)$ with coeffs. linear in $(r, s)$

• The intersection of the three heights is the vanishing of a 3x3 determinant, with linear entries in $(r,s)$.

• Given a cubic curve in $(r,s)$, it is identically zero iff it passes through a certain number of suitable distributed points in the $(r, s)$ plane.
Issues:

• Bounding the degree of the “bad” set.


• “Checking” procedure: internal dragging and verifying numerically / symbolically the validity of the statement at the given instance.
• Precision for checking numerically each instance
• Possibility to do it symbolically in GeoGebra
• Yet, “minor” epistemological obstacle. What if our paper / pencil proof is wrong? What if our computer has a bug?
• At least, different kind of obstacle from “probabilistic” checking.
Towards an i-GeometryBook:

• OMNISCIENT GeoGebra
• As a teacher... 😊

What could be the role of knowing geometric properties if a simple, widely accessible, free tool can automatically

• FIND
• VERIFY
• DISCOVER

such properties well beyond our personal ability?

What is the pedagogical role of knowing facts?
Didactical Issues: the role of reasoning and proving

- As an auxiliary tool, what opportunities, what differences involve using GeoGebra ART?

- Can guide student exploration, provide hints, answer (partially) questions...

- Can help building up diagrams, locus...

- Helping teachers!
• Math proofs: an identity feature.

• Trained when the output of reasoning can be automatically obtained?

• We can teach neither as nor the mathematics of the past....

• Not, particularly, in the case of Euclidean geometry, which is so much tool driven

• The message is clear: classical proof must move over...


From doing things better to doing better things?


- QED Manifesto (1994): “QED is ... a project to build a computer system that effectively represents all important mathematical knowledge and techniques.”
  
  https://www.cs.ru.nl/~freek/qed/qed.html,
  https://en.wikipedia.org/wiki/QED_manifesto
PLANE GEOMETRY:

AN ELEMENTARY TEXTBOOK

BY

SHALOSH B. EKHAD, XIV

(CIRCA 2050)

DOWNLOAD FROM THE FUTURE

BY

DORON ZEILBERGER

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